Financial optimization

Edited by

STAVROS A. ZENIOS

CAMBRIDGE
UNIVERSITY PRESS

PUBLISHED BY THE PRESS SYNDICATE OF THE UNIVERSITY OF CAMBRIDGE
The Pitt Building, Trumpington Street, Cambridge, United Kingdom

CAMBRIDGE UNIVERSITY PRESS
The Edinburgh Building, Cambridge CB2 2RU, UK www.cup.cam.ac.uk
40 West 20th Street, New York, NY 10011–4211, USA www.cup.org
10 Stamford Road, Oakleigh, Melbourne 3166, Australia
Ruiz de Alarcón 13, 28014 Madrid, Spain

First published 1993
Reprinted 1995, 1997, 1999

A catalogue record for this book is available from the British Library

Library of Congress Cataloguing in Publication Data applied for

ISBN 0 521 41905 0 hardback

Transferred to digital printing 2002

Contents

1 Some financial optimization models: I Risk management

HENRIK DAHL, ALEXANDER MEERAUS and STAVROS A. ZENIOS

1 Introduction

Since the early seventies the domain of financial operations witnessed a significant transformation. The breakdown of the Bretton Woods Agreement, coupled with a liberalization of the financial markets and the inflation and oil crisis of the same time, led to increased volatility of interest rates. The environment of fixed-income securities, where private and corporate investors, insurance, and pension fund managers would turn for secure investments, became more volatile than the stock market. The fluctuation of bonds increased sharply after October 1979 when the Federal Reserve Bank adopted a policy allowing wider moves in short-term interest rates. According to the volatility indexes, compiled by Shearson Lehman Economics, bonds were more volatile than stocks by a factor of seven in the early eighties.

Uncertainty breeds creativity, but so does a dynamic market where intelligent answers to complex problems are rewarded immediately. As a result we have seen an increased use of advanced analytic techniques in the form of optimization models for many diverse aspects of financial operations. Several theoretical developments provided the building blocks on which an analyst could base a comprehensive planning model. Models for the estimation of the term structure of interest rates, the celebrated Black–Scholes formula for valuating options, and other complex instruments, were added to the long list of contributions since Markowitz's seminal work on mean-variance analysis for stock returns in 1952.

During the same period tools from management science/operations research reached a stage of maturity and sophistication that gained the attention of practitioners in this dynamic environment. Operations research analysts found a very exciting problem domain where their tools could have a significant impact. Developments in computing technology, with the advent of workstations, facilitated the easy development and validation of models. As a result optimization models are becoming indispensable tools in several domains of financial operations. It is probably too early to pass judgment but *financial optimization* promises to be an area of applications comparable to the use of management science models in logistics, transportation, and manufacturing.

In this chapter we hope to demystify several widely used optimization models. The field of financial optimization has reached a stage where the appropriate modeling techniques are well understood for most applications. Nevertheless, problems remain where either new modeling approaches are required or existing solution techniques are not appropriate. Even in the better understood models, however, knowledge of the management science and the finance communities about them remains anecdotal. We provide here a description of some key optimization models. Financial optimization models are classified into two broad areas of application: (1) *risk management* and (2) *financial engineering*. Within each class we discuss several models and point out how these models have a common underlying theme (often, but not always). For each application we provide a brief description of the problem with references to related literature, we define the underlying optimization model in its most basic form, and we discuss important variants or extensions. Quite often we point at open problems and difficulties either in modeling the problem or in solving the model. Our goal is to help analysts understand the applications, by removing much of the jargon which is usually encountered in this field. We also aim at helping users understand the models and remove the "black-box" syndrome from some very important analytic techniques.

This chapter is organized as follows: section 2 classifies the problem domains of financial optimization and section 3 discusses several models that relate to risk management. The companion chapter in this book presents models of financial engineering and provides a brief survey of current solution methodologies. A technical report by the same authors, Dahl, Meeraus, and Zenios (1989) provides a library of well-documented financial optimization models and data, developed in the general algebraic modeling system GAMS of Brooke, Kendrick, and Meeraus (1988).

2 Problem domains

Most financial optimization models may be classified in two broad classes according to their primary objective. These classes we call here: (1) *risk management* and (2) *financial engineering*. Risk-management models are used to select portfolios with specified exposure to different risks. Financial-engineering models are used to structure new financial instruments in order to target specific investor preferences, or to take advantage of arbitrage opportunities. Below we shall characterize both classes and describe the scope of optimization within each one.

2.1 Risk management

One primary function of financial markets is to transfer risk. The transfer mechanism assigns market prices to each type of risk – called *risk premia* – at which supplies and demands are equalized. Although this means that in equilibrium all risks are priced fairly, so all securities have the same expected instantaneous rate of return, one cannot infer that all securities are equally good for all investors. Thus, due to the nature of their business, some investors prefer current income over future performance, while others are concerned with always staying fully funded. Investors who are willing to take a bet on their market views can do so by taking on certain risks, while those who are uncertain or risk averse can hedge their positions.

Risk management is concerned, firstly, with selecting which risks one is to be exposed to and which risks to be immunized against. Secondly, it is concerned with assessing the risks of different securities, and, thirdly, with the construction and maintenance of portfolios with the specified risk-return characteristics. The focus of optimization models is primarily on the third activity, but all three are integrated and interdependent. We give here a classification of different financial risks and examine methods of risk control. Emphasis is placed on the role of optimization models for risk management.

2.1.1 Financial risks

Financial risk is multidimensional. Therefore, a prerequisite to the selection of risk exposure is the identification of the risk forms that are present. The following list serves as the general framework for our discussion:

1 Market risk,
2 Shape risk,
3 Volatility risk,
4 Sector risk,
5 Currency risk,
6 Credit risk,
7 Liquidity risk,
8 Residual risk.

Market risk has slightly different interpretations, depending on which market is analyzed. In the stock market this form of risk is associated with the movements in the market index of portfolio returns. According to the Capital Asset Pricing Model (CAPM), see for example Sharpe (1985), all securities must be priced so that their expected returns at equilibrium are a linear combination of the risk-free return and the market index portfolio return. The weight of the latter for a particular security is the security beta (β) which indicates the relative marginal variation of the returns of that security with respect to the market portfolio. Hence, when the market moves by 1% the expected return on a security will move by β%.

In the fixed-income market, the traditional measure of market risk is interest-rate risk. In general terms this is risk caused by movements in the overall level of interest rates on straight, default-free securities. More specifically, it is the risk associated with a uniform increase in all default-free interest rates. When interest rates rise marginally the price of a regular bond drops by the bond's dollar duration, viz. the first-order derivative of the yield–price relation. Measured as percentage changes, the price effect is given by the bond duration, viz. the elasticity of the yield–price relation for that bond.

CAPM in the stock market, and duration models in the fixed-income market, are single-factor models of security returns (i.e., both assume a single source of risk: the market). Security attributes, like bond cashflows, determine the security sensitivity to movements in this risk factor. Hence, when a risk premium for the factor is determined by general equilibrium conditions, security returns are also determined as the product of price (risk premium) and quantity (contents of the factor).

In more general models one encounters several independent risk factors. In this context security attributes determine the sensitivity of the security to each factor. At equilibrium, the total supply and demand for each risk must be equal. Each risk factor has a risk premium associated with it and, at equilibrium, security returns are determined as the sum over all factors of the total value of that factor in the security (premium times quantity of the factor). This hypothesis is termed the Arbitrage Pricing Theory, Sharpe (1985). Under this hypothesis, market risk is merely the effect of one out of many risk factors, the effect of which is measured by the conventional beta (for stocks) or duration (for bonds).

Shape risk is applicable to the fixed-income market. It is the risk caused by non-parallel shifts of interest rates on straight, default-free securities (i.e., changes in the shape of the term structure of interest rates). To see the effect of shape risk assume that the yield curve is initially flat at 10%. The prices of two zero coupon bonds, one maturing in one year and the other maturing in ten years, are therefore 90.91 and 38.55, respectively. The two securities have durations of 1 and 10, respectively, so for parallel shifts of the term structure, the price of the second security is expected to move roughly ten times as much as the price of the first one, and in the same direction. An investor who expects interest rates to decline would therefore prefer the ten-year bond which promises the highest return. However, if the yield curve tilts upward so the one-year rate decreases to 9% and the long rate increases to 11%, the investor will experience a loss of 9.56% relative to investing in the one-year bond, being subjected to shape risk.

Shape risk can be quantified. Empirical analyses of the returns on U.S. treasuries indicate that three independent factors are sufficient to explain 98% of the variations of the term structure; see Garbade (1986) and Litterman *et al.* (1988). These factors are characterized by their shape impact: changes in the first factor imply almost parallel shifts to the term structure, so this could be thought of as the market risk factor. Changes in the second factor steepens the curve, while changes in the third factor implies changes in the overall curvature of the curve. Furthermore, these factors have been historically stable. This implies that one can compute the return sensitivity of different instruments to each factor and be reasonably certain to quantify shape risk. Having quantified the risk allows investors to select securities that expose them to risk according to their views as to parallel movements, steepening of the term structure, and so on.

Volatility risk is most clearly displayed in options. These instruments are characterized by highly asymmetric returns. If the underlying instrument is worth more than the strike price of a call option on the expiration date, the option is exercised returning the difference. But, if the underlying security is worth less than the strike price, the option expires worthless. Thus, an option resembles an insurance policy: it has value only if chances are that something might happen. The more volatile the markets are, the higher is the price of an option. Therefore, volatility changes have a major impact on options and securities embedding options (for example callable bonds, Mortgage-Backed Securities (MBS) which include prepayment options, etc.), even in an environment that is unchanged in all other respects.

Volatility risk is not only present in options. Regular straight bonds are also subject to volatility risk. The reason is that the yield–price relation is convex. This property implies that bond prices are affected more by a unit yield drop than by a unit yield increase. Therefore, the higher the volatility of yields around a common expected value, the higher the bond expected return. The effect is more important the more convex the yield–price relation is, i.e., the longer the bond maturity and the more dispersed the bond cashflows are. Similarly, the sensitivity of options to volatility is derived from their high positive convexity. Thus, sensitivity to volatility may be approximated by convexity.

Sector risk stems from events affecting the performance of a group of securities as a whole. A sector is a set of securities sharing some common characteristics. Thus, the treasury sector consists of those securities issued by the US Treasury. The agency sector covers bonds issued by various government agencies like the Federal Home Loan Bank, the Government National Mortgage Association, and so on. Mortgage-backed securities may also be defined as a sector.

Since sectors share some common attributes, they are likely to be influenced by common risk factors. Consider for instance the mortgage sector: when interest rates decline, volatility increases, and macroeconomic factors are "favorable", the likelihood of prepayment increases. An investor who expects this scenario to happen may choose to take a bet on his views by selling MBS.

Currency risk is the risk caused by exchange-rate fluctuations. Investors who own portfolios in foreign currency denominated securities will lose when exchange rates depreciate and gain when they appreciate. Another type of risk in international investment is political, or country, risk. Governments may change tax policy, trade policy, or even expropriate foreign investments.

Credit risk covers risks due to up- or down-grading of a borrower's credit worthiness. These changes are caused by changing prospects on the issuer's ability to meet all future obligations. Thus, if a borrower is more likely to default on some or all future payments, his credit worthiness deteriorates, and investors in turn will demand a higher premium for holding the debt. This in turn implies a price drop, hence risk. Credit risk is of importance when considering corporate bonds, but it is also a major influence on corporate money market instruments, bonds issued by sovereigns, etc.

Bonds exposed to credit risk can be thought of as contingent claims. Thus credit risk is related to the shape and volatility risk factors. However, other factors relating to the management and activities of the issuing firm contribute to credit risk (e.g., variations in earnings, age, debt/equity ratio, and so on). Bond ratings are designed to indicate credit worthiness, and controlling portfolio composition across ratings may therefore be used as a means to partial credit risk protection. A comprehensive approach demands analysis of the individual firm to see if its particular circumstances justify taking a bet on its credit worthiness.

Liquidity risk is due to the possibility that the bid–ask spread on security transactions may change. This type of risk is especially important for actively managed portfolios which depend on frequent trading. If the liquidity of a particular instrument worsens (i.e., the bid–ask spread widens) losses will materialize when selling the security if all other market conditions are unaltered. There exist many and diverse reasons for liquidity drops. For instance, due to prepayment on mortgages the circulating volume of a MBS may drop, making it more difficult to match buy and sell orders. Another reason could be institutional changes in the market place. For instance one of the side effects of the recent Exchange Reform in Denmark has been a marked reduction in liquidity for large groups of securities, as can be witnessed from the daily published bond price list from the Copenhagen Exchange. Typical measures of the liquidity of a security are circulating volume and trade volume over a period.

Residual – or specific – risk is, as the name indicates, all other risk. In as much as the previous list accounts for systematic influences, residual risk is security specific and non-systematic. Much of the activity to "beat the market" lies in buying securities which are expected to be underpriced, thus representing relative value, and selling seemingly overpriced securities. In effect, this means taking a position on residual risk.

2.1.2 Risk control techniques

An important question for controlling risk is "How systematic is a particular type of risk (i.e., to what degree does it affect all securities in a given sector)?" Non-systematic risks, that result in returns with correlation close to zero across instruments, can be reduced by diversification. Diversification, however, only leads to risk averaging for highly correlated risks. In the latter case hedging strategies are required. Distinguishing systematic from non-systematic risk is important in order to develop the appropriate investment strategy.

To see how hedging works, consider the following single-factor model. Assume that the risk factor evolves according to some stochastic differential equation (i.e., an Itô process, Ritchken (1987)). Let:

F be the factor level,

t represent time,

μ and σ be two deterministic functions depending on time and the factor level, and ω represent a Gauss–Wiener process.

Then the factor evolves according to:

$$dF = \mu dt + \sigma d\omega \tag{1}$$

Now, assume that the price P of a given security is a twice continuously differentiable function of the single factor and of time. Then, by Itô's lemma, the security price evolves according to:

$$dP = \frac{\partial P}{\partial F} dF + \frac{\partial P}{\partial t} dt + \frac{1}{2} \frac{\partial^2 P}{\partial F^2} dF^2 \tag{2}$$

Substituting (1) into (2) and collecting terms yields:

$$dP = \left(\mu \frac{\partial P}{\partial F} + \frac{\partial P}{\partial t} + \frac{1}{2} \sigma^2 \frac{\partial^2 P}{\partial F^2} \right) dt + \sigma \frac{\partial P}{\partial F} d\omega \tag{3}$$

The first term at the right-hand side of equation (3) is deterministic and depends only on time. It is therefore risk free and represents the time value of the security. The second term, however, is stochastic. It represents the impact of random shocks to the underlying factor on the security price.

To develop a simple hedge, thus eliminating the factor risk, consider two securities exposed to the same factor with prices P and Q respectively. Choose nominal values in the two securities, x and y respectively, such that:

$$x\sigma \frac{\partial P}{\partial F} d\omega + y\sigma \frac{\partial Q}{\partial F} d\omega = 0$$

i.e.:

$$x \frac{\partial P}{\partial F} + y \frac{\partial Q}{\partial F} = 0 \tag{4}$$

Then the overall position is risk free. Equation (4) states that the factor dollar duration of the position must be zero to eliminate factor risk: $\frac{\partial P}{\partial F}$ is the sensitivity of the security price to marginal changes in the factor level, or the security's *factor loading*. This hedging arrangement works out only if the factor is common to both securities (i.e., it is systematic), so that marginal returns are perfectly correlated. It is easily seen that, if each of the two securities were also exposed to residual factors, the hedging arrangement in (4) would have eliminated the common factor risk but left the residual risks in the portfolio.

The investment strategy $\{x,y\}$ derived from (4) is only a local hedge. In general $\frac{\partial P}{\partial F}$ is time dependent and to ensure that the portfolio remains riskless the position must be continuously adjusted. In practice, this is not feasible, and the problem is solved by discrete portfolio rebalancing. This calls for stabilizing conditions on the hedge. Having eliminated first-order effects by (4) stabilization is done by forcing second-order, or convexity, restrictions to improve tracking. We could also view this problem from a different angle: By not adjusting the hedge continuously, the portfolio is subject to volatility risk. As described earlier volatility risk is controlled by convexity conditions.

Note that the principles behind the single-factor hedge in equations (1)–(4) apply to multifactor models as well. For instance, in the presence of two common factors three securities would suffice to eliminate risk. Two would be used to eliminate the first factor and the third would be combined with the other two to eliminate the second factor.

Dropping the non-descript term "factor" and assuming that F is bond yield then (4) is the classical duration matching strategy, see Zipkin (1989). Immunization against interest-rate risk is achieved when dollar durations on assets and liabilities are equal, (i.e., net dollar duration is zero). Another interpretation is reached by letting F represent the market. Then, in the single-factor world of CAPM a risk-free position is achieved when net sensitivity to the market is zero (i.e., the net portfolio beta is zero). In other words, equation (4) describes the principles of hedging market risk.

2.1.3 The scope for optimization in risk management

Viewed in the context of the previous section, risk management is nothing more than taking positions in generic or specific attributes of the securities. An important problem, however, is that generic attributes are not traded in the market! Rather, one can invest in securities which are in effect packages of attributes. This complicates risk management. For instance, suppose that we wish to target a particular exposure to interest-rate risk because we expect a parallel downward shift of the yield curve. If we could simply buy a pure duration bond (i.e., one exposed to interest-rate risk and nothing else) risk management would be simple. But such a bond is not traded. Instead we can buy a real bond which is simultaneously exposed to interest-rate risk and other factors. Targeting duration using one such bond may inadvertently increase shape risk, volatility risk, credit risk and so on. This means that comprehensive risk management is faced with the problem of simultaneously controlling the interaction of many securities and their attributes in shaping overall portfolio exposure.

The problem is complicated even further when realizing that most investors face several institutional requirements in setting up a portfolio. Thus, regulations or firm policy may limit holdings in particular sectors, accounting rules may imply bounds on holdings in securities traded at a discount or premium, and the resources allocated to risk management may favor more conservative approaches over active management, putting yet other constraints on the portfolio composition.

The setting in which the problems are addressed – as outlined above – explains the important role of optimization models in risk management. Mathematical programming techniques can effectively identify the solution to complex portfolio planning problems with many constraints. Using mathematical programming we can find feasible solutions to the problem, or demonstrate that a risk exposure target is unattainable.

An important unresolved issue is: "What to optimize?" On the surface, this question seems trivial: minimize the cost of setting up the portfolio, or maximize its expected return. However, modern portfolio theory dictates that systematic risks are rewarded in a way that makes expected instantaneous returns equal on all securities in equilibrium, and all securities with the same attributes should have the same expected return to eliminate arbitrage. Therefore, if all systematic risk is eliminated, we are left with a risk-free position plus non-systematic risk. The expected rate of return on this position must be equal to the risk-free rate to eliminate arbitrage. The consequence is that high expected returns imply exposure to high levels of risk. Thus, maximizing returns or minimizing costs has the hidden property of maximizing uncontrolled risk.

Consider as an example the standard bond portfolio immunization model (explained in detail in section 3.1). The objective is to maximize net portfolio yield subject to present value and dollar duration constraints. The results of this model depend on the selection universe. If, for instance, low credit corporate bonds are included, the model will pick such bonds to achieve immunization, since they have higher yields than straight, default-free bonds. An apparent gain is realized by speculating in credit and sector risk without a systematic measure of the exposure to these two risk factors.

When the universe is restricted to current US Treasuries (i.e., bonds with no default risk, liquidity risk, etc.) the optimization results in a two-bond portfolio, consisting of a very long and a very short bond (a barbell). The overall portfolio is dollar duration matched, hence free of interest-rate risk. However, the portfolio is maximally exposed to shape risk. Also, to maintain dollar duration matching, the portfolio must be rebalanced relatively frequently, incurring high transaction costs and liquidity risks which are not accounted for in the optimization model.

To achieve higher stability and reduce shape risk, second-order constraints may be imposed on the model. The result will be less dispersed cashflows across the horizon and a reduction of risk. The position will still be exposed to volatility risk, however, and to correct, one may continue to add higher-order constraints. In the limit, the result will be a cashflow matched portfolio! This could have been obtained right away by using a cashflow matching model – for example, the dedication model given in section 3.3 – which apparently results in a more costly portfolio than the immunization model. However, as the example shows, the reason for this cost differential is that the dedicated portfolio is less risky than the immunized portfolio. Therefore, there is reason to expect that, over a long period, a dedicated portfolio will yield approximately the same as an immunized portfolio. Empirical confirmation of this observation is given in Maloney and Logue (1989).

The lesson drawn from the examples is that optimization can not be used blindly. Instead it must be coupled with a careful analysis of which risks the model is buying when maximizing returns. Thus, the role of the portfolio manager becomes crucial. The manager must be able to formulate clear objectives, state the economic and institutional constraints, choose a suitable selection universe, and analyze risks and returns. In all this, optimization is a tool which simplifies the shaping process. If all systematic risks are monitored, optimization can be used to pick underpriced securities and thus enhance performance. If risks are not monitored, however, optimization will maximize exposure to them. Optimization will make a good portfolio manager better, and a bad one worse!

2.2 Financial engineering

With the continued deregulation of markets, the increased volatility, and the intensified competition in the financial industry, financial innovation with the engineering of new instruments has accelerated rapidly. Well-known new products include standardized financial futures and options, floating-rate instruments, caps and floors, interest-rate and currency swaps, mortgage-backed securities, adjustable rate preferred stock, as well as derivatives of all these (for a general reference, see Walmsley (1988)).

Most of these products represent new packages of old attributes. It is well known, for example, that options can be replicated by a continuously rebalanced portfolio of a risk-free security and the underlying instrument. Nevertheless the new products have been successful additions to the financial markets. The main reason is that the new instruments typically carve out a few generic risk attributes from the initial building blocks, making it easier to control risk. Options, for example, are primarily volatility instruments but they are also exposed to interest-rate risk. A position where a call option is bought and a put option is sold, and where both options have common underlying instrument, expiration date, and strike price, is equivalent to a forward contract. However, breaking up the forward into options, and buying or selling these, enables investors to take direct positions in volatility, a possibility which the forward itself does not offer.

In this sense, financial engineering is one more aspect of risk management. As discussed above, risk management is a process of creating a portfolio of securities with certain attributes, from existing attribute packages. However, the resulting portfolio is not sold as a standardized product, while the objective of financial engineering is to design products that are added to the market.

A functional categorization of financial engineering products follows the list of financial risks of section 2.1.1. Thus, interest-rate futures and forwards, floating-rate notes and inverse floaters are targeted at isolating *market risk*. Interest-rate swaps can be thought of as *shape* instruments, since their return depends on changes to the shape of the term structure. Options are *volatility* products. Index-linked securities often carry *sector* attributes. Currency swaps, options, futures, and rolling forwards are examples of *currency risk* instruments. Other instruments are primarily credit enhancing. These include a long list of securitized assets (MBS, securitized bank loans, etc.) and asset swaps. Liquidity enhancement is inherent in put bonds and secondary-market mortgage products. Finally, stock and bond indices can be thought of as stripping out residual risk.

Reviewing this list, it becomes apparent that financial engineering is a mixture of service activity and arbitrage. By repackaging and stripping risk attributes to fit investor preferences and needs, the financial engineer improves the marketability of the products. In effect, he takes over some of the responsibilities of decentralized portfolio managers, and should be rewarded with a service fee. At the same time, however, the intention is to utilize mispricings in the market to capture risk premia. Thus, by stripping a US Treasury in zero coupon bonds the financial engineer disaggregates shape risk. If he can sell the resulting portfolio for more than the price of the treasury, he captures a riskless arbitrage profit which is due to mispriced shape risk.

In order to standardize products, certain guidelines must normally be obeyed. Thus, to obtain an AAA rating on an issue, rating agencies may require proof that the structure is default free under both best-case and worst-case scenarios. To simplify this task, standard procedures and scenarios have been developed, which need not be completely in line with reality. For instance, when issuing Collateralized Mortgage Obligations (CMOs), one requirement is that the collateral must sustain bond retirements, both in the very unlikely case of an immediate complete prepayment of all mortgages in the collateral and in the equally unlikely case of no prepayment, Sykes (1987).

As in the case of risk management, the mere existence of such constraints make optimization a key tool in financial engineering. Mathematical programming efficiently ensures feasibility and identifies the difficulties when feasibility cannot be attained. However, there also appears to be genuine opportunities for optimization in financial engineering. The reason is that there is generally some flexibility in structuring constraints, whether rating-agency or market dictated. For example, when structuring CMOs, the expected life of a tranche determines its price. A three-year tranche is priced using a spread over three-year treasuries. However, three years in market terms means any time period in the interval (3.0, 3.4) years. This slack may be sufficient reason to optimize profits, if several structures are feasible. Similarly, with markets not always in equilibrium, certain attributes may be relatively cheaper in some securities than in others, which again implies a scope for constrained profit maximization or cost minimization.

3 Model domains

In this section we discuss some optimization models used in risk-management applications. For each model, we briefly describe its theoretical background and then present the basic components of a mathematical prototype. Also, when relevant, we discuss potential or actual extensions of the models. We establish first some common notation. Unless stated otherwise in subsequent sections the following are used:

$U = \{1,2,3, \ldots, I\}$ denotes the universe of securities,
$i \in U$ indicates a security from the universe,
$T = \{1,2,3, \ldots, T_{max}\}$ denotes a set of discrete points in time,
$t \in T$ indicates a point in time,
C_{it} indicates the cashflow from security $i \in U$ at time $t \in T$,
x_i is the nominal holdings of security $i \in U$.

3.1 A bond portfolio immunization model

Immunization is a portfolio strategy used to match interest-rate risk of an asset portfolio against a future stream of liabilities, in order to achieve net zero market exposure. There is a large literature on portfolio immunization, see for instance Bierwag (1987), Fabozzi and Pollack (1987), Granito (1984), and Platt (1986). We describe here the fundamentals.

3.1.1 Background

Portfolio immunization is in essence a hedging strategy based on the principles of section 2.1. As both assets and liabilities are interest-rate sensitive (i.e., sensitive to the same common factor) a hedging strategy can be set up which eliminates net sensitivity to that factor. It was shown in section 2.1 that immunization is achieved when a portfolio is selected with net zero present value sensitivity to the factor of interest.

To compute the interest rate sensitivity of a cashflow, consider the yield–price relation. Let:

r_i denote the cashflow yield, and
P denote the present value.

The present value of a security is given by:

$$P_i = \sum_{t \in T} C_{it}(1 + r_i)^{-t} \tag{5}$$

Differentiating with respect to cashflow yield, gives the present value sensitivity k_i – or *dollar duration* – of the cashflow:

$$k_i = -\sum_{t \in T} tC_{it}(1 + r_i)^{-(t+1)} \tag{6}$$

It is seen that dollar duration is additive. Thus, the dollar duration of a portfolio is given by:

$$k_p = \sum_{i \in U} k_i x_i \tag{7}$$

Given the present value, P_L, and dollar duration, k_L, of liabilities, an immunized portfolio must satisfy the two conditions that present values and dollar durations on assets and liabilities be equal:

$$\sum_{i \in U} P_i x_i = P_L \tag{8}$$

$$\sum_{i \in U} k_i x_i = k_L \tag{9}$$

3.1.2 The optimization model

Immunized portfolios can be established in many ways. It is therefore natural to examine whether they can be put together optimally. The most commonly used objective has been to maximize the asset portfolio yield. The idea is that since the portfolio return is "risk-free" we might as well maximize it. The portfolio yield is given implicitly by equation (5) which is a non-linear expression. It turns out, however, that a first-order approximation to the true portfolio yield is the dollar duration weighted average yield of the individual securities in the portfolio, i.e.:

$$r \approx \frac{\Sigma_{i \in U} k_i r_i x_i}{\Sigma_{i \in U} k_i x_i} \tag{10}$$

The denominator in (10) is given by (9) to be equal to k_L, hence maximizing approximate portfolio yield is equivalent to simply maximizing the numerator in (10). In this case, the core immunization model is a linear programming problem:

[IMMUNIZATION1]

$$\underset{x \in \Re^I}{\text{Maximize}} \sum_{i \in U} k_i r_i x_i$$